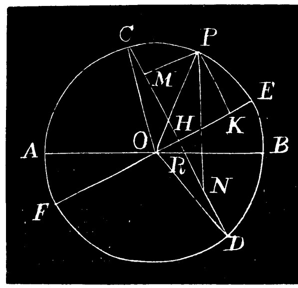
330. "Two points are taken at random within a circle on opposite sides of a given diameter, and a third point is taken at random in the circumference; find the average area of the triangle formed by joining the points."

SOLUTION BY PROF. E. B. SEITZ, KIRKSVILLE, MO.



Let $ACBD$ be the given circle, AB the given diameter, M, N two random points on opposite sides of AB, CD the chord through them, P a random point in the circumference, and O the center of the circle. Draw the diameter EF perpendicular to CD, and PK perpendicu'r to EF.

Let $OA = r$, $RM = x$, $RN = y$, $RC = u$, $RD = v$, $\angle COH = \theta$, $BOH = \varphi$, and $POE = \psi$. Then we have

$$u = r \sin\theta + r\cos\theta\tan\varphi, \qquad v = r\sin\theta - r\cos\theta\tan\varphi,$$

$$\text{area } MNP = \tfrac{1}{2}r(x+y)(\cos\psi - \cos\theta) = u_1, \text{ when } \psi < \theta,$$

and $$\text{area } MNP = \tfrac{1}{2}r(x+y)(\cos\theta - \cos\psi) = u_2, \text{ when } \varphi > \theta,$$

An element of surface at M is $r\sin\theta\, d\theta dx$, and at N it is $(x+y)d\varphi dy$, and an element of the circumference at P is $rd\psi$. The limits of θ are 0 and $\frac{1}{2}\pi$; of φ, $-\theta$ and θ, and doubled; of x, 0 and u; of y, 0 and v; and of ψ, 0 and θ, and θ and π.

By limiting P to the semi-circumference ECF, the whole number of ways the three points can be taken is $\frac{1}{4}\pi^2 r^4 . \pi r$; hence the required average is

$$\frac{8}{\pi^3 r^5}\int_0^{\frac{1}{2}\pi}\int_{-\theta}^{+\theta}\int_0^u\int_0^v\left\{\int_0^\theta u_1 r d\psi + \int_\theta^\pi u_2 r d\psi\right\} r\sin\theta\, d\theta d\varphi dx (x+y) dy$$

$$=\frac{4}{\pi^3 r^2}\int_0^{\frac{1}{2}\pi}\int_{-\theta}^{+\theta}\int_0^u\int_0^v (\pi - 2\theta + 2\tan\theta)\sin\theta\cos\theta d\theta d\varphi dx(x+y)^2 dy$$

$$=\frac{2r^2}{3\pi^3}\int_0^{\frac{1}{2}\pi}\int_{-\theta}^{+\theta}(\pi-2\theta+2\tan\theta)(7\tan^2\theta+\tan^2\varphi)(1-\cos^2\theta\sec^2\varphi)\sin\theta\cos^3\theta d\theta d\varphi$$

$$=\frac{4r^2}{9\pi^3}\int_0^{\frac{1}{2}\pi}[(\pi-2\theta)\cos\theta+2\sin\theta](24\theta\sin^2\theta-3\theta+3\sin\theta\cos\theta-22\sin^3\theta\cos\theta)\sin\theta d\theta$$

$$=\frac{r^2}{\pi}\left[\frac{13}{18}+\frac{352}{81\pi^2}\right].$$

SOLUTION OF MISCEL. PROB. (2), P. 149, VOL. VII, BY PROF. SCHEFFER.

If in the series $S = ax+bx^2+cx^3+dx^4+ex^5+\ldots$ we put $x = y \div (1+vy)$, we get $S = ay(1+vy)^{-1}+by^2(1+vy)^{-2}+cy^3(1+vy^{-3}+dy^4(1+vy)^{-4}+\ldots$

Expanding by the Binomial Theorem, we obtain

$$ay-(av-b)y^2+(av^2-2bv+c)y^3-(av^3-3bv^2+3cv-d)y^4+\ldots$$

Substituting for y its equiv't $x \div (1-vx)$, we have the req'd transformation.

[This poblem was solved in a similar manner by Prof. Kershner.]